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ON SOME PROPERTIES OF GROUPS OF p - COCHAINS[1]

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ABSTRACT

In this paper we give some properties of coboundary operator to abelian group of p-functions and prove inherited properties of coboundary operators to p-cochains.

Key words: p-function, coboundary operator, homomorphism, p-cochain, compact support

1. INTRODUCTION

Algebraic topology often deals with questions of finding algebraic invariants for some class of topological spaces that classify those spaces up to some useful equivalence relation, like homeomorphism or homotopy equivalence. Here we present a certain topics of algebraic topology, namely we introduce the abelian group $\Phi^p(X,G)$ of p-functions, and we show the properties of coboundary operator (homomorphism) acting on this group. Using the two subgroups of $\Phi^p(X,G)$ of p-functions with empty and compact support, $\Phi_0^p(X,G)$ and $\Phi_C^p(X,G)$ respectively, we define quotient groups of p-measurable cochains $C_C^p(X,G) = {\Phi_C^p(X,G)}/{\Phi_0^p(X,G)}$ and

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$C^p(X,G) = {\Phi^p(X,G)}\big/{\Phi_0^p(X,G)}$. We show that these groups inherit the properties of the group $\Phi^p(X,G)$ and we show the homomorphism induced by the coboundary operator.

2. DEFINITION OF GROUPS OF p-FUNCTIONS AND COBOUNDARY OPERATORS

Let X be an arbitrary set, p be an arbitrary integer, such that $p \geq 0$ and let G be an abelian group. We define ***p*-function *with values in G*** as a function of $p+1$ variables $\varphi(x_0, x_1, \cdots, x_p)$, where $x_i, i = 0, \cdots, p$ are elements of X, taking only finite number of different values in G.

Let $\Phi^p(X,G)$ be the set of all p-functions with values in G. It is easy to show that the set $\Phi^p(X,G)$ is abelian group with respect to coordinate-wise addition of functions.

We define homomorphism $\delta\colon \Phi^p(X,G) \to \Phi^{p+1}(X,G)$, for every $p \geq 0$ in the following way: for each $\varphi \in \Phi^p(X,G)$ the corresponding $\delta\varphi \in \Phi^{p+1}(X,G)$ is such that:

$$(\delta\varphi)(x_0, x_1, \cdots, x_p, x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i \varphi(x_0, x_1, \cdots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_{p+1}).$$

"^" assigns that we drop the marked coordinate. The function defined this way is a finite one.

We will show that the above defined mapping is indeed a homomorphism.

If are φ, ψ given elements of $\Phi^p(X,G)$, then for arbitrary $x_0, x_1, \cdots, x_p, x_{p+1}$ from X we have:

$$\begin{aligned}
&\big(\delta(\varphi+\psi)\big)(x_0, x_1, \cdots, x_p, x_{p+1}) = \\
&\sum_{i=0}^{p+1} (-1)^i (\varphi+\psi)(x_0, x_1, \cdots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_{p+1}) = \\
&\sum_{i=0}^{p+1} (-1)^i \Big(\varphi(x_0, x_1, \cdots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_{p+1}) \\
&\qquad + \psi(x_0, x_1, \cdots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_{p+1})\Big) = \\
&\sum_{i=0}^{p+1} \Big((-1)^i \varphi(x_0, x_1, \cdots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_{p+1}) \\
&\qquad + (-1)^i \psi(x_0, x_1, \cdots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_{p+1})\Big) =
\end{aligned}$$

$$\sum_{i=0}^{p+1}(-1)^i(\varphi)\left(x_0, x_1, \cdots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_{p+1}\right)$$
$$+\sum_{i=0}^{p+1}(-1)^i(\psi)\left(x_0, x_1, \cdots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_{p+1}\right) =$$

$$(\delta\varphi)\left(x_0, x_1, \cdots, x_p, x_{p+1}\right) + (\delta\psi)\left(x_0, x_1, \cdots, x_p, x_{p+1}\right) =$$
$$(\delta\phi + \delta\psi)\left(x_0, x_1, \cdots, x_p, x_{p+1}\right).$$

From arbitrariness of $x_0, x_1, \cdots, x_p, x_{p+1}$ from X it follows that $\delta(\varphi+\psi) = \delta\varphi + \delta\psi$. We have proven that δ is a homomorphism.

Lemma 2.1 $\delta(\delta\varphi) = 0$ for arbitrary element φ from $\Phi^p(X, G)$.

Proof: Let φ be an arbitrary element from $\Phi^p(X, G)$. Then for arbitrary $x_0, x_1, \cdots, x_p, x_{p+1}$ from X the following is valid:

$$(\delta\varphi)\left(x_0, x_1, \cdots, x_p, x_{p+1}\right) = \sum_{i=0}^{p+1}(-1)^i\varphi\left(x_0, x_1, \cdots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_{p+1}\right).$$

Applying once more the operator δ, we get:

$$\left(\delta(\delta\varphi)\right)\left(x_0, x_1, \cdots, x_p, x_{p+1}\right) =$$
$$\delta\left(\sum_{i=0}^{p+1}(-1)^i\varphi\left(x_0, x_1, \cdots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_{p+1}\right)\right) =$$
$$\sum_{\boldsymbol{i=0, j\neq i}}^{\boldsymbol{p+1}}(-1)^i\left(\sum_{i=0}^{p+1}(-1)^i\varphi\left(x_0, x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{p+1}\right)\right)$$

The last sum contains the following pairs of summands:

$$(-1)^i(-1)^{j-1}\varphi\left(x_0, x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{p+1}\right)$$
$$(-1)^j(-1)^i\,\varphi\left(x_0, x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{p+1}\right),$$ when $i < j$

or

$$(-1)^i(-1)^j\varphi\left(x_0, x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{p+1}\right)$$
$$(-1)^j(-1)^{i-1}\,\varphi\left(x_0, x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{p+1}\right),$$ when $j < i$

that are in the same double sum and hence are canceling each other. Arbitrariness of the sequence $x_0, x_1, \cdots, x_p, x_{p+1}$ gives $\delta(\delta\varphi) = 0$. ♦

We call the operator δ a ***coboundary operator.***

3. PROPERTIES OF THE SUPPORT OF p-FUNCTION

From now on we will consider X to be a topological space.

The ***support of p-function*** is the subset $|\varphi|$ of Xgiven by:

$x \notin |\varphi|$ if and only if there is a neighborhood V of x, such that $\varphi(x_0, x_1, \cdots, x_p) = 0$, for arbitrary $x_0, x_1, \cdots, x_p \in V$, or

$x \in |\varphi|$ if and only if for each neighborhood V of x, there exist $x_0, x_1, \cdots, x_p \in V$, such that $\varphi(x_0, x_1, \cdots, x_p) \neq 0$.

The following properties are valid for the support of p-function:

Property 3.1 For arbitrary element φ of $\Phi^p(X, G)$, $|\varphi|$ is a closed subset of X.

Proof: Let $x \notin |\varphi|$, i.e., $x \in X \backslash |\varphi|$. From the definition of support $|\varphi|$ we have that there exists a neighborhood V of x, such that

$\varphi(x_0, x_1, \cdots, x_p) = 0$, for all $x_0, x_1, \cdots, x_p$ од V.

Let $y \in V$ be an arbitrary point of V. Then for this point and it's neighborhood V, it is valid

$\varphi(x_0, x_1, \cdots, x_p) = 0$, for all $x_0, x_1, \cdots, x_p$ from V, i.e., $y \in X \backslash |\varphi|$. Hence, $V \subset X \backslash |\varphi|$. Arbitrariness of point $x \in X \backslash |\varphi|$, yields that $X \backslash |\varphi|$ is an open set. It follows that $|\varphi|$ is closed as a complement of an open set since obviously $|\varphi| = X \backslash (X \backslash |\varphi|)$.♦

Property 3.2 $|O| = \emptyset$, for the zero p-function $O \in \Phi^p(X, G)$.

Proof: Let $x \in X$ be arbitrary. Let V be an open neighborhood of x. Without loss of generality, let's assume that $V = X$. For arbitrarily chosen $x_0, x_1, \cdots, x_p \in X$ we have $O(x_0, x_1, \cdots, x_p) = 0$. It follows that $x \notin |O|$. Hence we have $|O| = \emptyset$. ♦

Property 3.3 $|\varphi \pm \psi| \subset |\varphi| \cup |\psi|$ is valid for any φ, ψ in $\Phi^p(X, G)$.

Proof: Let $x \notin |\varphi| \cup |\psi|$. Then $x \notin |\varphi|$ and $x \notin |\psi|$. From the definition of the support of p-function, there exist open neighborhoods V and U of x, such that $\varphi(x_0, x_1, \cdots, x_p) = 0$ и $\psi(x_0, x_1, \cdots, x_p) = 0$ for arbitrary $x_0, x_1, \cdots, x_p$ from V and U. Then for the neighborhood $U \cap V$ of x we get

$$(\varphi \pm \psi)(x_0, x_1, \cdots, x_p) = \varphi(x_0, x_1, \cdots, x_p) \pm \psi(x_0, x_1, \cdots, x_p) = 0 \pm 0 = 0.$$

It follows that there exists a neighborhood $W = U \cap V$ of x such that $(\varphi \pm \psi)(x_0, x_1, \cdots, x_p) = 0$, for any $x_0, x_1, \cdots, x_p$ from W, i.e.,

$x \notin |\varphi \pm \psi|$. Hence we get $|\varphi \pm \psi| \subset |\varphi| \cup |\psi|$. ♦

Property 3.4 $|\delta(\varphi)| \subset |\varphi|$, for any $\varphi \in \Phi^p(X,G)$.

Proof: Let $\varphi \in \Phi^p(X,G)$ and let $x \in |\delta(\varphi)|$ be any point. Then for any neighborhood V of x, there exist points $x_0, x_1, \cdots, x_p, x_{p+1}$ such that

$$\delta(\varphi)(x_0, x_1, \cdots, x_p, x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i \varphi(x_0, x_1, \cdots, \hat{x}_i, \cdots, x_p, x_{p+1}) \neq 0.$$

It follows that there exists a subsequence $y_0, y_1, \cdots, y_p$ of a sequence $x_0, x_1, \cdots, x_p, x_{p+1}$ such that $\varphi(y_0, y_1, \cdots, y_p) \neq 0$ (otherwise $\delta(\varphi)(x_0, x_1, \cdots, x_p, x_{p+1}) = 0$ which contradicts our assumption). From the definition of the support of the p-function we have $x \in |\varphi|$. Hence $|\delta(\varphi)| \subset |\varphi|$. ♦

Note 3.1 It must be noted that an empty support is not an exclusiveness of the zero functions. Nonzero function $\varphi \in \Phi^p(X,G)$, can also have an empty support if it satisfies some conditions. Such conditions are given by the following:

Property 3.5 Given nonzero p-function $\varphi \in \Phi^p(X,G)$, has an empty support, i.e., $|\varphi| = \emptyset$, if and only if there exists an open cover $\boldsymbol{\mathcal{U}}$ of the topological space X, such that $\varphi(x_0, x_1, \cdots, x_p) = 0$, for any $U \in \boldsymbol{\mathcal{U}}$, and for any $x_0, x_1, \cdots, x_p$ from U.

Proof: Let there be an open cover $\boldsymbol{\mathcal{U}}$ of topological space X, such that for any $U \in \boldsymbol{\mathcal{U}}$, and any $x_0, x_1, \cdots, x_p$ from U, $\varphi(x_0, x_1, \cdots, x_p) = 0$ is valid. Then, for every $x \in X$, there is a $U \in \boldsymbol{\mathcal{U}}$, such that $x \in U$, and for every $x_0, x_1, \cdots, x_p$ from U, $\varphi(x_0, x_1, \cdots, x_p) = 0$. But this means that $x \notin |\varphi|$. It follows that $X \subseteq X \backslash |\varphi|$, hence $X = X \backslash |\varphi|$, i.e., $|\varphi| = \emptyset$.

Opposite, let $|\varphi| = \emptyset$. Then for every point $x \in X$, there is an open neighborhood $U_x \in \boldsymbol{\mathcal{U}}$, such that for any $x_0, x_1, \cdots, x_p$ from U_x, $\varphi(x_0, x_1, \cdots, x_p) = 0$ is valid. Now we consider the open cover $\boldsymbol{\mathcal{U}}$ to be the family of sets $\boldsymbol{\mathcal{U}} = \{U_x | x \in X\}$. This cover has the property that was required. ♦

Next we define

$$\Phi_0^p(X,G) = \{\varphi | \varphi \in \Phi^p(X,G), |\varphi| = \emptyset\} \text{ и}$$
$$\Phi_C^p(X,G) = \{\varphi | \varphi \in \Phi^p(X,G), |\varphi| = compact\}.$$

As a direct consequence of the previous properties we get

Property 3.6 Let X be a topological space. Then the following statements hold true:

(i) $\Phi_0^p(X,G) \subset \Phi_C^p(X,G)$

(ii) $\Phi_0^p(X,G), \Phi_C^p(X,G)$ are subgroups of $\Phi^p(X,G)$

(iii) $\delta\left(\Phi_0^p(X,G)\right) \subseteq \Phi_0^{p+1}(X,G)$

$\delta\left(\Phi_C^p(X,G)\right) \subseteq \Phi_C^{p+1}(X,G)$

Proof: (i) Let $\varphi \in \Phi_0^p(X,G)$ be any element. Then $|\varphi| = \emptyset$, and we know that the empty set is a compact set in any topological space, hence $\varphi \in \Phi_C^p(X,G)$. Arbitrariness of φ yields $\Phi_0^p(X,G) \subset \Phi_C^p(X,G)$.

(ii) Let φ, ψ be any elements in $\Phi_0^p(X,G)$. Then $|\varphi| = \emptyset$ and $|\psi| = \emptyset$. Property 3.3 yields $|\varphi \pm \psi| \subset |\varphi| \cup |\psi| = \emptyset$, i.e., $|\varphi \pm \psi| = \emptyset$. But this means that $\varphi \pm \psi \in \Phi_0^p(X,G)$, i.e., $\Phi_0^p(X,G)$ is a subgroup of $\Phi^p(X,G)$.

Let φ, ψ be any elements of $\Phi_C^p(X,G)$.Then $|\varphi|$ and $|\psi|$ are compact sets and so is the set $|\varphi| \cup |\psi|$. Again by using the Property 3.3, we have $|\varphi + \psi| \subset |\varphi| \cup |\psi|$, which means that $|\varphi + \psi|$ is also a compact set, as a closed subset of a compact set. It follows that $\varphi + \psi$ is an element of $\Phi_C^p(X,G)$, i.e., $\Phi_C^p(X,G)$ is a subset of $\Phi^p(X,G)$.

(iii) Let $\varphi \in \Phi_0^p(X,G)$. Then $|\varphi| = \emptyset$ and from Property 3.5, there is an open neighborhood $\boldsymbol{\mathcal{U}}$ of the topological space X, such that $\varphi(x_0, x_1, \cdots, x_p) = 0$, for any $U \in \boldsymbol{\mathcal{U}}$ and for any $x_0, x_1, \cdots, x_p$ од U. Then $U \in \boldsymbol{\mathcal{U}}$ and any $x_0, x_1, \cdots, x_p$ from U the following statement is valid

$$(\delta\varphi)(x_0, x_1, \cdots, x_p, x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i \varphi(x_0, x_1, \cdots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_{p+1}) = 0.$$

But this means that $(\delta\varphi)(x_0, x_1, \cdots, x_p, x_{p+1}) = 0$ for any $U \in \boldsymbol{\mathcal{U}}$ and any $x_0, x_1, \cdots, x_p$ from U, i.e., $\delta\varphi \in \Phi_0^{p+1}(X,G)$.

Let $\varphi \in \Phi_C^p(X,G)$. Then $|\varphi|$ is a compact set, so from Property 3.4, we have that $|\delta(\varphi)| \subset |\varphi|$. This and the closeness of $|\delta(\varphi)|$ means that $|\delta(\varphi)|$ is a compact set. Thus $\delta\varphi \in \Phi_C^{p+1}(X,G)$. ♦

4. PROPERTIES OF THE SUPPORT OF p-MEASURABLE COCHAIN

In order to give the definition of a p-measurable cochain we define the following quotient groups:

$$C_C^p(X,G) = {\Phi_C^p(X,G)}\Big/{\Phi_0^p(X,G)}$$

$$C^p(X,G) = {\Phi^p(X,G)}\Big/{\Phi_0^p(X,G)}.$$

The element of the quotient group $C^p(X,G)$ is called a ***p*-measurable cochain** of space X, ***with coefficients in the group G.***
The element of the quotient group $C_C^p(X,G)$ is called a ***p*-measurable cochain** of space X, ***with compact support and coefficients in the group G***.

In the following we will show that in analogy to the definition of the support of a given p-function, we can define the support of the corresponding p-measurable cochain. Then we will prove the generalization to Properties 3.1 to 3.4, i.e., it will be shown that the quotient groups $C^p(X,G)$ and $C_C^p(X,G)$ inherit some of the properties of the groups $\Phi^p(X,G)$ and $\Phi_C^p(X,G)$.

Lemma 4.1 For any $\boldsymbol{u} \in C^p(X,G)$ and any $\varphi,\psi \in \boldsymbol{u}$,

$$|\varphi| = |\psi|.$$

Proof: Let $\varphi,\psi \in \boldsymbol{u}$, for $\boldsymbol{u} \in C^p(X,G)$. Then $\varphi - \psi \in \Phi_0^p(X,G)$, i.e., there is an element $h \in \Phi_0^p(X,G)$, such that $\varphi - \psi = h$. But then $\varphi = \psi + h$ and from Property 3.3 we have that $|\varphi| = |\psi + h| \subset |\psi| \cup |h|$. Now since $h \in \Phi_0^p(X,G)$ we have that $|h| = \emptyset$ and so

$$|\varphi| \subset |\psi| \qquad (1)$$

The symmetry of the equivalence relation used to define the quotient group $C^p(X,G)$, yields

$$|\psi| \subset |\varphi| \qquad (2)$$

From (1) and (2) it follows that $|\varphi| = |\psi|$ for any elements $\varphi,\psi \in \boldsymbol{u}$, for $\boldsymbol{u} \in C^p(X,G)$. Thus, without loss of generality we can define $|\boldsymbol{u}|$, for any given class of equivalence $\boldsymbol{u} \in C^p(X,G)$. We will call the set $|\boldsymbol{u}|$ ***the support of the p-measurable cochain.***

This definition of the support of a equivalence class allows us to generalize Properties 3.1 to 3.4.

Property 4.1 Given an equivalence class $\boldsymbol{u} \in C^p(X,G)$, the set $|\boldsymbol{u}|$ is closed in topological space X.

Proof: Let $\boldsymbol{u} \in C^p(X,G)$ be any equivalence class. Following the definition of support of p-measurable cochain $|\boldsymbol{u}| = |\varphi|$, for any $\varphi \in \boldsymbol{u}$. But then the closeness of the set $|\boldsymbol{u}|$ follows immediately from the closeness of $|\varphi|$ in the topological space X. ♦

Property 4.2 Let X be a topological space and $\boldsymbol{u} = O \in C^p(X,G)$. Then $|\boldsymbol{u}| = \emptyset$. Moreover $|\boldsymbol{u}| = \emptyset$ if and only if $\boldsymbol{u} = O$.

Proof: Let $\boldsymbol{u} = O \in C^p(X,G)$. Then $|\boldsymbol{u}| = |\varphi| = \emptyset$, for any $\varphi \in \Phi_0^p(X,G)$.

If $|\boldsymbol{u}| = \emptyset$, than for every $\varphi \in \boldsymbol{u}$ we have $|\varphi| = \emptyset$, i.e., $\varphi \in \Phi_0^p(X,G)$. This means that $\boldsymbol{u} \subset \Phi_0^p(X,G)$. Now in the opposite direction: if $\varphi \in \Phi_0^p(X,G)$ then$|\varphi| = \emptyset$ and for every $\psi \in \boldsymbol{u}$ we have $|\varphi - \psi| = \emptyset$. This means that $\varphi - \psi \in \Phi_0^p(X,G)$, i.e., $\varphi \in \boldsymbol{u}$ and $\Phi_0^p(X,G) \subset \boldsymbol{u}$. But this means that $\boldsymbol{u} = O = \Phi_0^p(X,G)$. ♦

Property 4.3 For any $\boldsymbol{u}, \boldsymbol{v} \in C^p(X,G)$ the following statement is valid:

$$|\boldsymbol{u} \pm \boldsymbol{v}| \subset |\boldsymbol{u}| \cup |\boldsymbol{v}|.$$

Proof: Let $\boldsymbol{u}, \boldsymbol{v} \in C^p(X,G)$ and let $\varphi \in \boldsymbol{u}$, $\psi \in \boldsymbol{v}$ be arbitrarily chosen elements of the corresponding equivalence class. Then by definition we have $|\boldsymbol{u}| = |\varphi|, |\boldsymbol{v}| = |\psi|$ and $|\boldsymbol{u} \pm \boldsymbol{v}| = |\varphi \pm \psi|$.Property 3.3 then gives $|\boldsymbol{u} \pm \boldsymbol{v}| = |\varphi \pm \psi| \subset |\varphi| \cup |\psi| = |\boldsymbol{u}| \cup |\boldsymbol{v}|$. ♦

Property 4.4 Let $\boldsymbol{u} \in C^p(X,G)$ be any given element. Then $|\delta(\boldsymbol{u})| \subset |\boldsymbol{u}|$.

Proof: Let $\boldsymbol{u} \in C^p(X,G)$ be any given element. Let $\varphi \in \boldsymbol{u}$ be any representative of the given equivalence class. Then we have, $|\boldsymbol{u}| = |\varphi|$ and $|\delta(\boldsymbol{u})| = |\delta(\varphi)|$ since the representative of the class $\delta(\boldsymbol{u})$ is $\delta(\varphi)$. Property 4.4 then gives us $|\delta(\boldsymbol{u})| = |\delta(\varphi)| \subset |\varphi| = |\boldsymbol{u}|$. Thus $|\delta(\boldsymbol{u})| \subset |\boldsymbol{u}|$. ♦

Proposition 4.5 The operator δ induces a homeomorphisms

$$\delta: C^p(X,G) \to C^{p+1}(X,G)$$

$$\delta: \mathrm{C}_\mathrm{C}^p(X,G) \to \mathrm{C}_C^{p+1}(X,G).$$

Proof: It is enough to show that the group $\mathrm{C}_\mathrm{C}^p(X,G)$ is a subgroup of the group $C^p(X,G)$. Indeed, if $\boldsymbol{u}, \boldsymbol{v} \in \mathrm{C}_\mathrm{C}^p(X,G)$, then any given representatives $\varphi \in \boldsymbol{u}, \psi \in \boldsymbol{v}$ are elements of $\Phi_\mathrm{C}^p(X,G)$ and consequently the sets $|\varphi|, |\psi|$ are compact. Since $|\boldsymbol{u}| = |\varphi|$, $|\boldsymbol{v}| = |\psi|$ we have $|\boldsymbol{u} + \boldsymbol{v}| \subset$

$|\boldsymbol{u}| \cup |\boldsymbol{v}|$. This means that the equivalence class $\boldsymbol{u}+\boldsymbol{v}$ has a compact support as a closed subset of the compact set $|\boldsymbol{u}| \cup |\boldsymbol{v}|$.

Hence $C_C^p(X,G) \subset C^p(X,G)$ and is closed with respect to the group operation, which means that $C_C^p(X,G)$ is a subgroup of the group $C^p(X,G)$. Now the induced homeomorphisms $\delta: C^p(X,G) \to C^{p+1}(X,G)$ and $\delta: C_C^p(X,G) \to C_C^{p+1}(X,G)$ follow naturally from Property 3.6.

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